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Generation of Frames

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It is well known that, given a generic frame, there exists a unique frame operator which satisfies, together with its adjoint, a double operator inequality. In this paper we start considering the inverse problem, that is how to associate a frame to certain operators satisfying the same kind of inequality. The main motivation of our analysis is the possibility of using frame theory in the discussion of some aspects of the quantum time evolution, both for open and for closed physical systems.

KEY WORDS: frames; Hilbert spaces, quantum time evolution.

1. INTRODUCTION

Whenever we deal with a (separable) Hilbert space $\mathcal{H}$ the first problem we usually face with is the way in which an arbitrary element $f \in \mathcal{H}$ can be conveniently expressed. As we know the usual choice is to expand f in terms of an orthonormal (o.n.) basis $\{e_n\}$ of $\mathcal{H}$: in this way the expansion is particularly simple, $f = \Sigma_n \langle e_n, f\rangle e_n$, and the Parseval equality holds, $\Sigma_n |\langle e_n, f\rangle|^2 = \| f\|$.

Sometimes, however, the conditions of our (mathematical or physical) problem force us to consider a set of vectors $\{\Phi_n\}$ which is no longer o.n. but is still a basis of $\mathcal{H}$. We have a typical example of this situation when the set $\{\Phi_n\}$ forms a Riesz basis, see Cohen *et al.* (1992) and references therein, that is a set of vectors such that

– no Φ_{n_0} lies within the closure of the finite linear span of the other Φ_n, and
– $\exists A > 0,\ B < \infty$ so that, for any $f \in \mathcal{H}$,

$$A\| f\|^2 \le \sum_n |\langle \Phi_n, f\rangle|^2 \le B\| f\|^2. \qquad (1.1)$$

This last property implies that the vectors of the set $\{\Phi_n\}$ generates the whole Hilbert space, while the first condition says that these vectors are linearly independent.

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Many examples moreover also exist of sets of vectors which are not Riesz bases but still have a relevant role in the description of $\mathcal{H}$. For instance, if we consider any overcomplete set of coherent states (Klander and Streater, 1985), this set satisfies a relation similar to the one in (1.1), but the vectors are not linearly independent. Equation (1.1) is also satisfied by some sets of wavelets, (Daubechies, 1992). Sets of vectors of this kind are known as frames. In other words, we can say that a frame is a set of generators of $\mathcal{H}$, but, since the vectors are not independent, the way in which a vector $f \in \mathcal{H}$ can be expanded in terms of these vectors is, in general, not unique.

Many mathematical properties of the frames have been discussed in the literature, see Ali *et al.* (2000) and Casazza (2000) for an overview. In reference (Bagarello, 1997), the construction of different frames starting from a fixed one, and the construction of a "faster" perturbation scheme has been considered. Here two questions were raised: is it possible to "reverse" the procedure which, given a frame $\mathcal{I}$, produces a unique frame operator $F_{\mathcal{I}}$? And, if this can be done, is this frame unique?

The second question is the following: in Bagarello (1997) we have shown how to construct a (1,1)-frame starting from an (A, B)-frame. The role of the frame bounds was crucial in order not to have a trivial result. In fact, if we follow the procedure developed in Bagarello (1997), every (1,1)-frame can only produce itself. So it is natural to wonder whether there exists some different way to obtain an (A, B)-frame starting from a (1, 1)-frame, for some fixed positive numbers A and B. We will be more precise in the next section.

In this paper we answer to both these questions in a satisfactorily and natural way. This is the content of the next two sections, respectively related to self-adjoint and non-self-adjoint *generating operators*, that is, roughly speaking, bounded operators in the Hilbert space, satisfying a certain operatorial inequality which *produce* frames.

In our examples, contained in Sections 3 and 4, we will consider with particular attention the problem of the stability of a given frame under quantum mechanical time evolution, both for open and for closed systems, as well as other aspects related to quantum mechanics.

2. NOTATION AND KNOWN RESULTS

In this section we will recall some known results about frames in order to keep the paper self-contained and to introduce our notation. Most of these results can be found in Daubechies (1992), Bagarello (1997), and Daubechies (1990).

Let $\mathcal{H}$ be a Hilbert space and J a given set of indexes. Let also $\mathcal{I} \equiv \{\varphi_n, n \in J\}$, be a set of vectors of $\mathcal{H}$. We say that $\mathcal{I}$ is an (A,B)-*frame* of $\mathcal{H}$ if there exist two positive constants, called frame bounds, $0 < A \leq B < \infty$, such that the inequalities

$$A\|f\|^2 \leq \sum_n |\langle \varphi_n, f\rangle|^2 \leq B\|f\|^2 \tag{2.1}$$

hold for any $f \in \mathcal{H}$.

To any such set $\mathcal{I}$ can be associated a bounded operator $F_{\mathcal{I}} : \mathcal{H} \to l^2(J) = \{\{c_n\}_{n\in J} : \Sigma_{n\in J}|c_n|^2\} < \infty$ defined by the formula

$$\forall f \in \mathcal{H} \quad (F_{\mathcal{I}} f)_j = \langle \varphi_j, f\rangle. \tag{2.2}$$

We will omit the dependence on $\mathcal{I}$ of F in the following. Equation (2.1) implies that $\|F\| \leq \sqrt{B}$, so that F is bounded. The adjoint of the operator F, $F^\dagger$, which maps $l^2(J)$ into $\mathcal{H}$, is such that

$$\forall\{c\} \in l^2(J) \quad F^\dagger c \equiv \sum_{i\in J} c_i\varphi_i \tag{2.3}$$

Condition (2.1) can be rewritten in the following equivalent way:

$$A\mathbb{I} \leq F^\dagger F \leq B\mathbb{I} \tag{2.4}$$

which must be understood in the sense of the operators (Reed and Simon, 1980). We have used $\mathbb{I}$ to identify the identity operator in $B(\mathcal{H})$.

Condition (2.4) implies that the operator $(F^\dagger F)^{-1}$, exists and is still bounded in $\mathcal{H}$. In other terms, both $F^\dagger F$ and $(F^\dagger F)^{-1}$ belong to $B(\mathcal{H})$.

Following the literature, see Daubechies (1992) for instance, one defines the *dual frame* of $\mathcal{I}$, $\tilde{\mathcal{I}}$, as the set of vectors $\tilde{\varphi}_i$ defined by

$$\tilde{\varphi}_i \equiv (F^\dagger F)^{-1}\varphi_i \quad \forall i \in J. \tag{2.5}$$

In particular (Daubechies, 1992), $\tilde{\mathcal{I}}$ is a $(\frac{1}{B}, \frac{1}{A})$-frame. Defining now a new operator between $\mathcal{H}$ and $l^2(J)$ as $\tilde{F} \equiv F(F^\dagger F)^{-1}$, it is easy to prove that $\tilde{F}$ is such that $(\tilde{F} f)_i = \langle\tilde{\varphi}_i, f\rangle$, for all $f \in \mathcal{H}$. Moreover, the following relations hold: $\tilde{F}^\dagger F = F^\dagger\tilde{F} = \mathbb{I}$. These equalities produce the following reconstruction formulas:

$$f = \sum_{i\in J}\langle\varphi_i, f\rangle\tilde{\varphi}_i = \sum_{i\in J}\langle\tilde{\varphi}_i, f\rangle\varphi_i \tag{2.6}$$

for all $f \in \mathcal{H}$ (Daubechies, 1992). In (Daubechies, 1992) it is also discussed that, since $\varphi_i = \tilde{\tilde{\varphi}}_i$ for all $i \in J$, then the dual frame of the set $\tilde{\mathcal{I}}$ is nothing but the set $\mathcal{I}$ itself.

A generalization of this procedure has been proposed in Bagarello (1997): let us define the operator

$$\mathcal{F}_1 \equiv F^\dagger F. \tag{2.7}$$

The norm of this operator is bounded from above and from below, $A \leq \|\mathcal{F}_1\| \leq B$, $\mathcal{F}_1$ is positive, $\mathcal{F}_1 \geq 0$, self-adjoint, $\mathcal{F}_1 = \mathcal{F}_1^\dagger$, and its action on a given vector of $\mathcal{H}$ is given by $\mathcal{F}_1 f = \Sigma_{i\in J}\langle\varphi_i, f\rangle\varphi_i$.

Let E_λ be the family of spectral operators of $\mathcal{F}_1$. We can write, making use of the spectral theorem.

$$\langle\mathcal{F}_1\Phi, \Psi\rangle = \int_A^B \lambda d\langle E_\lambda\Phi, \Psi\rangle, \quad \forall\Phi, \Psi \in \mathcal{H}. \tag{2.8}$$

Because of the fact that $0 < A \leq B < \infty$, we can define arbitrary powers, positive and negative, of the operator $\mathcal{F}_1$:

$$\langle\mathcal{F}_\alpha\Phi, \Psi\rangle \equiv \langle(F^*F)^\alpha\Phi, \Psi\rangle = \int_A^B \lambda^\alpha d\langle E_\lambda\Phi, \Psi\rangle, \tag{2.9}$$

for all $\Phi, \Psi \in \mathcal{H}$, and $\forall\alpha \in \boldsymbol{R}$. This implies that

$$A^\gamma\mathbb{I} \leq \mathcal{F}_\gamma \leq B^\gamma\mathbb{I} \quad \forall\gamma < 0 \tag{2.10}$$

$$B^\gamma\mathbb{I} \leq \mathcal{F}_\gamma \leq A^\gamma\mathbb{I} \quad \forall\gamma \geq 0. \tag{2.11}$$

Given an arbitrary real number α let us define the following vectors:

$$\varphi_i^{(\alpha)} \equiv \mathcal{F}_\alpha\varphi_i \qquad \forall i \in J, \tag{2.12}$$

and let us call $\mathcal{I}^{(\alpha)}$ the set of these vectors. In Bagarello (1997) we proved that all these sets are frames in $\mathcal{H}$. In particular $\mathcal{I}^{(\alpha)}$ is an $(A^{2\alpha+1}, B^{2\alpha+1})$-frame if $\alpha > -\frac{1}{2}$, is a (1, 1)-frame if $\alpha = -\frac{1}{2}$, and is a $(B^{2\alpha+1}, A^{2\alpha+1})$-frame if $\alpha < -\frac{1}{2}$.

As it is clear, this procedure produces a tight frame with frame bounds equal to 1 starting from a generic frame. Of course this does not imply that the vectors $\varphi_i^{(-1/2)}$ form an o.n. basis since normalization of these vectors is not ensured.

The reconstruction formula (2.6) can now be generalized in the following way (Bagarello, 1997): for any $f \in \mathcal{H}$ and for any real α, we have

$$f = \sum_{i\in J}\left\langle\varphi_i^{(\alpha)}, f\right\rangle\varphi_i^{(-1-\alpha)} \tag{2.13}$$

We refer to Bagarello (1997) for further comments and examples. Here we want to be more precise about the two problems already mentioned in Section 1. The first one is the following: given a frame, we know that it gives rise to a single operator $\mathcal{F}_1 = F^\dagger F$. Now, given a self-adjoint operator $\mathcal{F}_1$ which satisfies an inequality like (2.4), is it possible to associate to this operator a single frame? In Bagarello (1997) we constructed a counterexample for a finite-dimensional Hilbert space, showing that this frame, if it exists, is not necessarily unique. We will see in the next section that also for infinite-dimensional Hilbert spaces the uniqueness is not guaranteed, while the construction of such a frame can be easily undertaken. Our result will appear as a concrete application to the framework developed in

Bagarello (1997) of the fact that any bounded, surjective operator applied on an o.n. basis or to a frame still yields a frame.

The second open problem is closely related to the previous one: given an (A, B)-frame $\{\varphi_i\}$ and its related operator $\mathcal{F}_1$, we find a unique (1, 1)-frame $\{\varphi_i^{(-1/2)}\}$ obtained as in (2.12). Now, given a (1, 1)-frame $\{\phi_i\}$, is it possible to find a (unique) (A, B)-frame $\{\Psi_i\}$ such that $\phi_i = \Psi_i^{(-1/2)}$? Again the answer is in general negative and this can be shown simply by giving an example of (1, 1)-frame which can be obtained by different nontight frames (Bagarello, 1997). However, these different frames are not completely unrelated among them, and in the next section we will show, among other things, which kind of relations do exist between the frame operators associated to the frames $\{\Psi_i\}$ and $\{\phi_i\}$.

3. REVERSING THE PROCEDURE

In this section we will consider the following question: given a self-adjoint operator $Z : \mathcal{H} \longrightarrow \mathcal{H}$ such that two positive constants A and B, $0 < A \leq B < \infty$ exist satisfying

$$A\mathbb{I} \leq Z \leq B\mathbb{I}, \tag{3.1}$$

is it possible to define one (or more) frame in some sense *related* to Z?

Of course if we do not specify what has to be meant by *related*, this question has no much meaning. First of all we can observe already at this stage that uniqueness is not very reasonable, since if an (A, B)-frame can be constructed starting from Z, then an entire family of frames can be easily generated simply following the procedure proposed in the previous section and in reference (Bagarello, 1997), at least if $A \neq 1$ and $B \neq 1$. It is clear, then, that all these frames are, in some sense, related to the operator Z.

In what follows we will propose a very sharp procedure to generate frames starting from the operator Z above *and* from another given frame or from a given o.n. basis. For this reason we speak of *generation of frames*. The main property of Z which will be used in the following is the possibility of defining, via spectral theorem, Z^α for any real value of α.

It is worth mentioning that part of our results, and more specifically the first part of Proposition 1 below, are close to those in Casazza (2000), where the notion of preframe operator has been introduced and analyzed.

We begin with the following Proposition, where we split the statement in two in order to stress the differences in the proof when considering an o.n. basis or simply a frame.

Proposition 1. *Let Z be a self-adjoint operator $Z : \mathcal{H} \longrightarrow \mathcal{H}$ such that inequality (3.1) holds for a given pair (A, B) of strictly positive quantities,* $0 < A \leq B$

$< \infty$. Let $B = \{e_n, n \in J\}$ be an o.n. basis of $\mathcal{H}$ and $\mathcal{I} = \{\varphi_n, n \in J\}$ a (C, D)-frame, $0 < C \leq D < \infty$. Then: defining

$$\eta_n^{(\alpha)} := Z^\alpha e_n, \quad \forall n \in J, \forall \alpha \in \boldsymbol{R} \tag{3.2}$$

the set $\mathcal{I}_{\eta^{(\alpha)}} = \{\eta_n^{(\alpha)}, n \in J\}$ is an $(A^{2\alpha}, B^{2\alpha})$-frame if $\alpha \geq 0$, and a $(B^{2\alpha}, A^{2\alpha})$-frame if $\alpha < 0$.
Also, defining

$$\Phi_n^{(\alpha)} := Z^\alpha \varphi_n, \quad \forall n \in J, \forall \alpha \in \boldsymbol{R} \tag{3.3}$$

then the set $\mathcal{I}_{\Phi(\alpha)} = \{\Phi_n^{(\alpha)}, n \in J\}$ is an $(A^{2\alpha}C, B^{2\alpha}D)$-frame if $\alpha \geq 0$, and a $(B^{2\alpha}C, A^{2\alpha}D)$-frame if $\alpha < 0$.

Proof: The proof of the first statement is an easy consequence of the Parceval equality for $\mathcal{B}$: for all $f \in \mathcal{H}$ we have

$$\sum_{n\in I} \left|\left\langle f, \eta_n^{(\alpha)}\right\rangle\right|^2 = \sum_{n\in I} |\langle Z^\alpha f, e_n\rangle|^2 = \|Z^\alpha f\|^2 = \langle f, Z^{2\alpha} f\rangle.$$

The conclusion follows from the definition of frame and from inequalities (2.10) and (2.11) for Z.

The second statement requires more care, since the Parceval equality does not hold for frames. Let us call T the frame operator associated to $\mathcal{I}$, then T must satisfy the inequality $C\mathbb{I} \leq T^\dagger T \leq D\mathbb{I}$. Therefore we have, for all $f \in \mathcal{H}$,

$$\sum_{n\in J} \left|\left\langle f, \Phi_n^{(\alpha)}\right\rangle\right|^2 = \sum_{n\in J} |\langle f, Z^\alpha \varphi_n\rangle|^2 = \sum_{n\in J} |\langle Z^\alpha f, \varphi_n\rangle|^2 = \sum_{n\in J} |(T(Z^\alpha f))_n|^2$$
$$= \|(T(Z^\alpha f))\|^2 = \langle T(Z^\alpha f), T(Z^\alpha f)\rangle = \langle (Z^\alpha f), T^\dagger T(Z^\alpha f)\rangle,$$

which, using the bounds on $T^\dagger T$, gives

$$C\|Z^\alpha f\|^2 \leq \sum_{n\in J} \left|\left\langle f, \Phi_n^{(\alpha)}\right\rangle\right|^2 \leq D\|Z^\alpha f\|^2.$$

As before, the conclusion follows from the inequalities on Z^α, (2.10) and (2.11). □

Remarks

(1) Obviously, it is clear that the first statement is a simple consequence of the second one, since an o.n. basis is simply a (1, 1)-frame of normalized vectors. However, since the proofs above are significantly different, we have chosen to consider the two situations separately.
(2) Secondly, if we take $\alpha = 0$ above, it is clear that $Z^\alpha = \mathbb{I}$, so that it is obvious that the transformed sets coincide with the original ones and, in

fact, $\mathcal{I}_{\eta(0)}$ is a (1, 1)-frame of normalized vectors (i.e. again an o.n. basis), while $\mathcal{I}_{\Phi(0)}$ is again a (C, D)-frame. If $\alpha = \frac{1}{2}$, then the set $\mathcal{I}_{\eta(1/2)}$ is an (A, B)-frame. This, in a certain sense, reverses the approach sketched in the previous section where a (1, 1)-frame was built starting from a given (A, B)-frame. Here we are starting with a particular (1, 1)-frame, that is with an o.n. basis, and we construct an (A, B)-frame.

(3) Finally, this result extends, in a certain sense, the one in Bagarello (1997) since the operator Z which produces the (1, 1)-frame starting from $\mathcal{I}$ needs not to be $\mathcal{F}_1 = R^\dagger R$, R being a given frame operator.

Let now Z be a *generating operator*, that is an operator on $\mathcal{H}$ satisfying the hypotheses of Proposition 1. We can use Z to produce frames, starting from a given o.n. basis $\mathcal{B}$. Let $\eta_n^{(\alpha)}$ be as in (3.2). Proposition 1 ensures that the set $\{\eta_n^{(\alpha)}\}$ is a frame, whose frame bounds depend on the value of α. The standard procedure, therefore, allows us to associate to $\{\eta_n^{(\alpha)}\}$ a frame operator, X_α, defined as $(X_\alpha f)_n = \langle \eta_n^{(\alpha)}, f\rangle$, $f \in \mathcal{H}$. Its adjoint $X_\alpha^\dagger$, is defined as usual and we have $X_\alpha^\dagger X_\alpha f = \Sigma_{n\in J}\langle \eta_n^{(\alpha)}, f\rangle \eta_n^{(\alpha)}$. It is easy to deduce now a relation between Z^α and X_α. This relation is

$$\|X_\alpha f\|_{l^2}^2 = \|Z^\alpha f\|_{\mathcal{H}}^2, \quad \forall f \in \mathcal{H}, \tag{3.4}$$

and it follows from the following equalities:

$$\|X_\alpha f\|_{l^2}^2 = \langle X_\alpha f, X_\alpha f\rangle = \langle f, X_\alpha^\dagger X_\alpha f\rangle = \sum_{n\in J}\left|\left\langle \eta_n^{(\alpha)}, f\right\rangle\right|^2$$
$$= \sum_{n\in J}|\langle e_n, Z^\alpha f\rangle|^2 = \|Z^\alpha f\|_{\mathcal{H}}^2,$$

which can still be rewritten as $\|X_\alpha Z^{-\alpha} g\|_{l^2} = \|g\|_{\mathcal{H}}$ for all g in $\mathcal{H}$. Moreover, this equality implies that

$$\|X_\alpha Z^{-\alpha}\|_{B(\mathcal{H},l^2)} = 1, \tag{3.5}$$

where $\|\cdots\|_{B(\mathcal{H},l^2)}$ is the norm in the Banach space of the bounded operators mapping $\mathcal{H}$ into $l^2(J)$. Notice that, while Z is given a priori, and it is therefore independent of the basis $\mathcal{B}$, X_α depends on the choice of the o.n. basis originating the frame $\{\eta_n^{(\alpha)} = Z^\alpha e_n\}$. Nevertheless, equality (3.4) implies that $\|X_\alpha f\|$ turns out to be independent of the choice of $\mathcal{B}$, for any $f \in \mathcal{H}$.

The relation between the two operators X_α and Z^α can be further clarified by introducing an unitary map $U_\mathcal{B}$, which also depends on the o.n. basis $\mathcal{B}$ of $\mathcal{H}$, defined in the following way:

$$U_\mathcal{B} : \mathcal{H} \to l^2(J) \quad \text{such that, given}$$
$$f \in \mathcal{H},\ f = \sum_n f_n e_n,\ \text{then}\ U_\mathcal{B}(f) = \{f_n, n \in J\}. \tag{3.6}$$

Incidentally, from this definition we deduce that $U_{\mathcal{B}}$ is nothing but the adjoint of the preframe operator wrt the o.n. basis $\{e_n\}$.

It is easy to check now that X_α and Z^α are related by the following equation

$$X_\alpha = U_{\mathcal{B}} Z^\alpha. \tag{3.7}$$

Indeed we have, for any given $f \in \mathcal{H}$, $(X_\alpha f)_n = \langle \eta_n^{(\alpha)}, f \rangle = \langle e_n, Z^\alpha f \rangle$, while $(U_{\mathcal{B}} Z^\alpha f)_n = (U_{\mathcal{B}}(\Sigma_m \langle e_m, Z^\alpha f \rangle e_m))_n = \langle e_n, Z^\alpha f \rangle$, so that Eq. (3.7) follows.

Equality (3.4) is now simply a consequence of (3.7) and of the unitarity of the operator $U_{\mathcal{B}}$.

If we consider the same problem for a frame instead of an o.n. basis the situation is, at least formally, even simpler. In fact, in this case, there is no need for introducing the operator $U_{\mathcal{B}}$. The computation is rather direct: calling Y_α the frame operator associated to the set $\{\Phi_n^{(\alpha)}\}$ we get

$$(Y_\alpha f)_n = \left\langle \Phi_n^{(\alpha)}, f \right\rangle = \langle \varphi_n, Z^\alpha f \rangle = (F(Z^\alpha f))_n,$$

F being the frame operator of the set $\mathcal{I}$. Therefore, due to the arbitrariness of f, we get

$$Y_\alpha = F Z^\alpha. \tag{3.8}$$

Notice that, if $\mathcal{I}$ is an o.n. basis, then F is nothing but $U(\mathcal{I})$ so that the result in (3.7) is recovered.

Let us now consider some examples.

3.1. Example 1

In this first example, we will show how to produce explicitly a class of frame generators, and what this procedure gives in some explicit situations. Let Q be an orthogonal projection operator acting on $\mathcal{H}$. This means that

$$Q = Q^2 = Q^\dagger. \tag{3.9}$$

Let then A and B be two positive constants such that the usual inequality $0 < A \leq B < \propto$, is satisfied. We define an operator

$$X = A\mathbb{I} + (B - A)Q, \tag{3.10}$$

which is clearly a self-adjoint operator on $\mathcal{H}$. Since $X - A\mathbb{I} = (B - A)Q = (B - A)Q^\dagger Q$, it follows that, whenever $A \neq B$, $X - A\mathbb{I}$ is a positive operator. Analogously $B\mathbb{I} - X = (B - A)(\mathbb{I} - Q) = (B - A)(\mathbb{I} - Q)^\dagger(\mathbb{I} - Q)$ is a positive operator, under the same hypothesis on A and B (if $A = B$ everything is simpler since $X = A\mathbb{I}$ and the example becomes trivial). Therefore $A\mathbb{I} \leq X \leq B\mathbb{I}$ and X is a generating operator.

Producing orthogonal projection operators is not a problem: the easiest way consists in starting with an o.n. basis (which nothing has to do in general with the basis $\mathcal{B}$ of Proposition 1), and use this basis in the following canonical way. Let, for instance, $\mathcal{C} = \{h_n, n \in \boldsymbol{N}\}$ be such a basis. Let $J \subset \boldsymbol{N}$ be a (finite) set of indexes. We call Q_J the following (finite rank) operator: $Q_J f = \Sigma_{n\in J}\langle h_n, f\rangle h_n$, for any $f \in \mathcal{H}$. It is clear that Q_J is self-adjoint and that $Q_J = Q_J^2$, so that Q_J is an orthogonal projection. Let $X_J = A\mathbb{I} + (B - A)Q_J$. This is a frame generator. Notice that, if we take $\mathcal{B} = \mathcal{C}$ above, then we do not go too far:

$$H_n = X_J h_n = \begin{cases} Bh_n & \text{if } n \in J, \\ Ah_n & \text{if } n \notin J \end{cases}$$

This means that we still get an orthogonal set but we lose (in a trivial way) the normalization of the vectors. More interesting is the result if we apply X_J to an o.n. basis $\mathcal{B}$ different from $\mathcal{C}$. The resulting vectors of our frame are now

$$\mu_n = X_J e_n = e_n(1 + (B - A)\|Q_J e_n\|^2) + (B - A)\sum_{l\neq n}\langle e_l, Q_J e_n\rangle e_l. \quad (3.11)$$

Just to be concrete we now give an example of this construction for $\mathcal{H} = \mathcal{L}^2(\boldsymbol{R})$. We consider the following well known o.n. bases of $\mathcal{L}^2(\boldsymbol{R})$:

$$\mathcal{B}_1 = \{H_{j,k}(x) = 2^{-j/2}H(2^{-j}x - k), \qquad j, k \in \mathbb{Z}\},$$

where

$$H(x) = \begin{cases} 1 & \text{if } x \in \left[0, \frac{1}{2}\right], \\ -1 & \text{if } x \in \left[\frac{1}{2}, 1\right], \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\mathcal{B}_2 = \left\{\varphi_n(x) = \frac{1}{\sqrt{2^n n! \pi^{1/4}}} e^{-x^2/2} H_n(x),\ n \in \boldsymbol{N}_0\right\},$$

with $H_n(x) = (-1)^n e^{x^2}(\frac{d^n}{dx^n}e^{-x^2})$. The first example, $\mathcal{B}_1$, is the Haar o.n. basis arising in the multiresolution analysis of $\mathcal{L}^2(\boldsymbol{R})$ (Daubechier, 1992) while $\mathcal{B}_2$ is the o.n. set of eigenfunctions of the harmonic oscillator, as obtained in any elementary textbook of quantum mechanics. Let us now see first how $\mathcal{B}_2$ can be used to obtain a frame from $\mathcal{B}_1$. Then we will show the opposite.

In order to keep the construction simple, we give here only the simplest example, which is obtained by considering the simplest nontrivial set $J : J = \{0\}$. The extension to more elements is only a matter of computation. Let Q_0 be defined as $(Q_0 f)(x) = \langle\varphi_0, f\rangle\varphi_0(x)$, $f(x) \in \mathcal{L}^2(\boldsymbol{R})$. Then $X = A\mathbb{I} + (B - A)\langle\varphi_0, .\rangle\varphi_0(x)$, and

$$\tilde{H}_{j,k}(x) = XH_{j,k}(x) = AH_{j,k}(x) + (B - A)\langle\varphi_0, H_{j,k}\rangle\varphi_0(x).$$

It follows that:

if $A = B = 1$ then $\tilde{H}_{j,k}(x) = H_{j,k}(x)$;

if $A = B \neq 1$ then $\tilde{H}_{j,k}(x) = AH_{j,k}(x)$, so that only the normalization is changed;

if $A \neq B$ every $H_{j,k}(x)$ is modified by the action of X. However, with this simple choice of J, it is clear that this change reduces to an addictive contribution which is proportional to a single function, $\varphi_0(x)$, but with a constant which depends on j and k. This situation can be made more interesting simply by taking more elements in the definition of J.

Exchanging the role of $\mathcal{B}_1$ and $\mathcal{B}_2$ we can also check, for instance, that the set of functions

$$\tilde{\varphi}_n(x) = A\varphi_n(x) + (B - A)\langle H_{0,0}, \varphi_n\rangle H_{0,0}(x)$$

is a frame, and the same remarks as above still apply.

In recent literature the role of the time evolution of certain sets in the Hilbert space has been discussed in connection with quantum mechanical systems. In particular in Klauder (2001), Antoine *et al.* (2001), and Crawford (2000), among the others, the so-called temporally stable coherent states have been discussed. A temporally stable coherent state is a coherent state $\psi(\alpha)$ whose time evolution still yields a coherent state. Since coherent states are deeply related with frames, it is natural to consider the same kind of problem for frames, and this is the content of the next example where we will discuss an open quantum system and its time evolution, provided by a semigroup of bounded operators on a certain Hilbert space $\mathcal{H}$. The outcome will be that the time evolution of a frame is still a frame, even if the frame bounds are, in general, modified during the time evolution. We will consider a closed system, arising, for instance, in the analysis of low-temperature superconductivity, in subsection 3.3. The next example is related to a not unitary operator, typical of quantum open systems.

3.2. Example 2

Let a and $a^\dagger$ be two operators satisfying the CCR algebra $[a, a^\dagger] = \mathbb{I}$, and N be the closure of $N_0 = a^\dagger a$. It is well known that, calling φ_0 the vector annihiled by a, $a\varphi_0 = 0$, then the vectors $\varphi_n = \frac{(a^\dagger)^n}{\sqrt{n!}}\varphi_0$, $n \in \boldsymbol{N} \cup \{0\}$, are an o.n. basis of the Hilbert space $\mathcal{H}$. This is the typical algebraic structure behind any quantum harmonic oscillator. Let us now define the self-adjoint operator $L = \mathbb{I} + (N + \mathbb{I})^{-1}$. It is straightforward to check that each φ_n is an eigenstate of L with eigenvalue $\frac{n+2}{n+1}$. This implies that

$$\mathbb{I} \leq L \leq 2\mathbb{I} \tag{3.12}$$

Since L is bounded there is no problem in defining the family of bounded and self-adjoint operators $T_t = e^{Lt}$, with t a positive parameter which we can think of as the time. It is clear that φ_n is also an eigenstate of T_t with eigenvalue $e^{\frac{n+2}{n+1}t}$. Moreover T_t satisfies the following inequality:

$$e^t \mathbb{I} \leq T_t \leq e^{2t} \mathbb{I} \tag{3.13}$$

Let now $\Xi = \{\mu_n, n \in J\}$ be a fixed (C, D)-frame. We define $\Xi^{(1)} = \{\mu_n^{(1)} = L\mu_n, n \in J\}$, and $\Xi^{(2)}(t) = \{\mu_n^{(2)}(t) = T_t \mu_n, n \in J\}$. Since $\frac{d}{dt} T_t = L T_t = T_t L$, we have

$$\frac{d}{dt}\mu_n^{(2)}(t)|_{t=0} = L\mu_n = \mu_n^{(1)}, \tag{3.14}$$

which relates the two sets. Moreover, Proposition 1 states that both $\Xi^{(1)}$ and $\Xi^{(2)}$ are frames. In particular $\Xi^{(1)}$ is a $(C, 4D)$-frame, while $\Xi^{(2)}(t)$ is a (Ce^{2t}, De^{4t})-frame. Therefore the time evolution T_t maps a frame in another frame. It is worth remarking that the frame bounds of $\Xi^{(1)}$ are not given, as one could expect, by the time derivative of those of $\Xi^{(2)}(t)$ (for $t = 0$), even if Eq. (3.14) holds. Finally, it is clear that the time derivative of the set $\Xi^{(2)}(t)$, which is made up of vectors $L\mu_n^{(2)}(t) = T_t \mu_n^{(1)}$, is still a frame with bounds Ce^{2t} and $4De4t$.

3.3. Example 3

We discuss here in some details an example which is closely related to a nontrivial model proposed in Quantum Many-Body in order to explain the phase transition giving rise to superconductivity at low temperature. The details of the model are given in Buffet and Martin (1978) and Martin (1979). This is a discrete model of an open system in which the matter, described in terms of Pauli matrices, interacts with a fermionic background. The model is defined on a finite lattice which we take here, for sake of simplicity, to consist of a single site. We will remove this assumption at the end of our analysis. The algebra of the Pauli matrices, in our single-lattice model, is given by $[\sigma^+, \sigma^-] = \sigma^0$, $[\sigma^\pm, \sigma^0] = \mp 2\sigma^\pm$, while the *canonical anticommutation relations* for the fermionic operators are $\{a_i, a_j^\dagger\} = \delta_{ij}$, $\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0$, where i and j take values between 1 and N, N being the different modes of the reservoir. Following (Buffet and Martin, 1978; Martin, 1979) we define

$$H = H_0 + \lambda H_I = (H_s + H_r) + \lambda H_I,$$

where $H_s = \tilde{\epsilon}\sigma^0$, $H_r = \Sigma_{i=1}^N \epsilon_i a_i^\dagger a_i$ and $H_I = \sigma^+ a(f) + \sigma^- a^\dagger(\overline{f})$. Here $\overline{\epsilon}$ and λ are real constants, ϵ_i are all nonnegative, and we have used the following notation: $a(f) = \Sigma_{i=1}^N a_i\, f(i)$ and $a^\dagger(\overline{f}) = \Sigma_{i=1}^N a_i^\dagger\, \overline{f}(i)$. The function f is a test function. This model is exactly the one site version of the one discussed in Buffet and Martin (1978) and Martin (1979) with $g = 0$, i.e. neglecting the mean-field interaction of the matter.

The eigenstates of H_s are clearly

$$\Psi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \Psi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}: H_s\Psi_\gamma = E_\gamma \Psi_\gamma,$$

where $E_\gamma = \gamma\bar{\epsilon}$ and $\gamma = \pm$. The eigenstates of H_r are obtained acting with the *creation* operators $a_i^\dagger$ on the *vacuum state* φ_0, which is defined by the following property: $a_i\varphi_0 = 0$ for all $i = 1, \ldots, N$. We put $\varphi_{n_1,\ldots,n_N} = (a_1^\dagger)^{n_1} \cdots (a_N^\dagger)^{n_N}\varphi_0$, where each n_i can only be 0 or 1, and $H_r\varphi_{n_1,\ldots,n_N} = E_{n_1,\ldots,n_N}\varphi_{n_1,\ldots,n_N}$, where $E_{n_1,\ldots,n_N} = \Sigma_{i=1}^N \epsilon_i$, and the sum is restricted to those indexes j such that $n_j = 1$. It is clear that the Hilbert space of the matter is simply $\boldsymbol{C}^2$, while the Hilbert space of the reservoir, $\mathcal{H}_{fer}$, is the linear span of the set $\mathcal{F} = \{\varphi_{n_1,\ldots,n_N}$, such that $n_i = 0, 1 \forall i = 1, \ldots N\}$. We call $\mathcal{H} = \boldsymbol{C}^2 \otimes \mathcal{H}_{fer}$. It is clear that $\mathcal{H}$ is a finite-dimensional Hilbert space. It is also easy to prove, using canonical estimates, that H satisfies the following bound $\|H\| \leq B$, where $B = |\tilde{\epsilon}| + \Sigma_{i=1}^N \epsilon_i + 2|\lambda|\Sigma_{i=1}^N |f(i)|$. Since the spectrum of H, as well as the spectrum of H_0, is discrete, we deduce that $-B\mathbb{I} \leq H \leq B\mathbb{I}$. Therefore, for any positive δ, the operator $\tilde{H} = H + (B + \delta)\mathbb{I}$, which is completely equivalent to H from the point of view of the dynamics of the system, satisfies the following inequality: $\delta\mathbb{I} \leq \tilde{H} \leq (2B + \delta)\mathbb{I}$. Therefore $\tilde{H}$ can be used to modify the o.n. basis of $\mathcal{H}$ whose vectors are $\Phi_{(\gamma),n_1,\ldots,n_N} = \Psi_\gamma \otimes \varphi_{n_1,\ldots,n_N}$ to obtain different frames. In particular, if we put

$$\eta_{(\gamma),n_1,\ldots,n_N}^{(\alpha)} = \tilde{H}^\alpha \Phi_{(\gamma),n_1,\ldots,n_N},$$

α being any real number, then the set $\{\eta_{(\gamma),n_1,\ldots,n_N}^{(\alpha)}\}$ is an $(\delta^{2\alpha}, (2B + \delta)^{2\alpha})$-frame if $\alpha \geq 0$ and an $((2B + \delta)^{2\alpha}, \delta^{2\alpha})$-frame if $\alpha \leq 0$.

The same conclusion holds true when we odd more sites to our lattice, as soon as the background is kept fermionic. The situation requires more care in the thermodynamical limit ($N \to \infty$) or for a bosonic reservoir. The reason for this is that unbounded operators necessarily appear in the game, and the above estimates cannot hold any longer.

In order to be concrete we see now what happens in the simplest situation, that is when $N = 1$. In this case the dimension of $\mathcal{H}$ is 4 and an o.n. basis consists of the vectors $\Phi_{(+),0}$, $\Phi_{(+),1}$, $\Phi_{(-),0}$, and $\Phi_{(-),1}$. The action of $\tilde{H}$ on these vectors can easily be computed, and it gives the following vectors: $\eta_{(+),1}^{(1)} = (\tilde{\epsilon} + \epsilon_1 + B + \delta)$ $\Phi_{(+),1}$, $\eta_{(-),0}^{(1)} = (-\tilde{\epsilon} + B + \delta)\Phi_{(-),0}$, $\eta_{(+),0}^{(1)} = (\tilde{\epsilon} + B + \delta)\Phi_{(+),0} + \lambda\overline{f}(1)\Phi_{(-),1}$, and $\eta_{(-),1}^{(1)} = (-\tilde{\epsilon} + \epsilon_1 + B + \delta)\Phi_{(-),1} + \lambda f(1)\Phi_{(+),0}$, which, as we have seen, form a $(\delta^2, (2B + \delta)^2)$-frame in our Hilbert space.

We want to end this section with a remark concerning the possibility of extracting an o.n. basis from a given (A, B)-frame. Both Proposition 1 and the results discussed in Bagarello (1997), just to cite the results closest to our set up, teaches how to get a (1,1)-frame from a generic (A, B)-frame. But the normalization of the vectors of the new frame cannot be implemented without breaking, in general, the frame condition (1.1). However, the following Lemma can be proved and gives an interesting constraint.

Lemma. *Let $\mathcal{B} = \{\varphi_n, n \in J\}$ be an (A, B)-frame and Z a self-adjoint operator defined on $\mathcal{H}$ such that $A\mathbb{I} \leq Z \leq B\mathbb{I}$. A necessary condition for the set $\mathcal{B}^{(z)} = \{\varphi_n^{(z)} = Z^{-1/2}\varphi_n, n \in J\}$ being an o.n. basis is*

$$\sqrt{A} \leq \|\varphi_n\| \leq \sqrt{B}, \quad \forall n \in J \tag{3.15}$$

Proof: From Proposition 1 we deduce that $\mathcal{B}^{(Z)}$ is a $(1, 1)$-frame. To conclude that $\mathcal{B}^{(Z)}$ is also an o.n. set we still have to chek that each $\varphi_n^{(Z)}$ is normalized, that is that $\langle\varphi_n, Z^{-1}\varphi_n\rangle = 1$ for any $n \in J$. Using the hypothesis on Z this can be true only if $B^{-1}\|\varphi_n\|^2 \leq 1 \leq A^{-1}\|\varphi_n\|^2$, so that our statement follows. □

4. MORE FRAME GENERATORS

In this section we extend the procedure discussed before so to include unitary operators, which are not frame generators with the above definition since they cannot satisfy an operator inequality like the one in (3.1). The need for this extension follows from the very well known fact that any unitary operator maps an o.n. basis into another o.n. basis, and (A, B)-frames into (A, B)-frames. Moreover, unitary operators are also physically quite relevant since, for instance, they describe the time evolution of conservative (closed) quantum mechanical systems.

Let T be an operator which maps the Hilbert space $\mathcal{H}$ into itself and such that two real positive constants α and β exist, $0 < \alpha \leq \beta < \infty$ for which

$$\alpha\mathbb{I} \leq T^\dagger T \leq \beta\mathbb{I}. \tag{4.1}$$

Of course, even if (4.1) ensures us that $T^\dagger T$ is invertible, this is not necessarily true for the operator T by itself, since it only has to satisfy the bounds $\sqrt{\alpha} \leq \|T\| \leq \sqrt{\beta}$ which are not sufficient to guarantee the invertibility of T. Another obvious remark is that, even if condition (4.1) strongly resembles the usual inequality that a frame operator (and its adjoint) must satisfy, a big difference really exists since T is required to map $\mathcal{H}$ into $\mathcal{H}$, while the frame operator maps $\mathcal{H}$ into a different Hilbert space, $l^2(J)$.

We have the following

Proposition 2. *Let T be an operator $T : \mathcal{H} \to \mathcal{H}$ such that inequality (4.1) holds for a given pair (α, β) of strictly positive quantities, $0 < \alpha \leq \beta < \infty$. Let $\mathcal{B} = \{e_n, n \in J\}$ be an o.n. basis of $\mathcal{H}$ and $\mathcal{I} = \{\varphi_n, n \in J\}$ a (C, D)-frame, $0 < C \leq D < \infty$. Then:*

if we define

$$\eta_n := T^\dagger e_n, \quad \forall n \in J, \tag{4.2}$$

the set $\mathcal{B}_n = \{\eta_n, n \in J\}$ *is an* (α, β)*-frame.*

Also, defining

$$\Phi_n := T^\dagger \varphi_n, \quad \forall_n \in J, \tag{4.3}$$

then the set $\mathcal{I}_\Phi = \{\Phi_n, n \in J\}$ *is an* $(\alpha C, \beta D)$*-frame.*

The proofs of both these statements are quite similar to those of Proposition 1 and will be omitted here.

Even if T is not assumed to be invertible, since it is a bounded operator it is clear that T^l makes sense for any positive value of l. The extension of the above result to T^l is not straightforward but when the operator T is normal, $[T, T^\dagger] = 0$, which is surely the case if T is unitary. In this case, using the same notation as above, we get the following natural results, which extend what stated above:

the set of vectors $\eta_n^{(l)} := (T^\dagger)^l e_n$ is an (α^l, β^l)-frame;

the set of vectors $\Phi_n^{(l)} := (T^\dagger)^l \varphi_n$ is an $(\alpha^l C, \beta^l D)$-frame;

4.1. Example 4

We go back now to consider the time evolution for a closed quantum mechanical system $\mathcal{S}$, which is described by the Schrödinger equation:

$$i\frac{d}{dt}\Psi(t) = H\Psi(t),$$

Here $\Psi(t)$ is the wave function describing $\mathcal{S}$ at time t and H is its hamiltonian, that is the energy of $\mathcal{S}$, which is a self-adjoint operator. As an example, you could consider the operator H given in Subsection 3.3 of the previous example or its modified form $\tilde{H}$. It may happen already for very simple systems that H is an unbounded operator, so that H cannot satisfy any inequality like (3.1). This is not the case in Subsection 3.3, where the hamiltonian is bounded. Therefore, but for some particular form of the hamiltonian, we cannot produce frames using the hamiltonian directly. However, this is not the end of the story: if Ψ is the value of the wave function for $t = 0$ it is well known that, at least if H does not depend explicitly on t, the solution of the Schrödinger equation above can be written as $\Psi(t) = e^{-iHt}\Psi$, since the unitary operator e^{-iHt}, which can be defined both for bounded and unbounded hamiltonians, describes the time evolution of $\mathcal{S}$ in the so-called Schrödinger-representation. This operator clearly does not satisfy inequality (3.1) for any choice of A and B, since its spectrum lies on the unit circle $|\lambda| = 1$. We can conclude that, in general, both the hamiltonian and the *time evolution operator* e^{-iHt} do not fit into the scheme developed in the previous section. However, it is trivial to check that for any physical hamiltonian H, bounded or not, e^{-iHt} satisfies the hypotheses of Proposition 2 above, since $(e^{-iHt})^\dagger e^{-iHt} = \mathbb{I}$, so that it can be used, for instance, to construct new frames starting from a given one.

For instance, let now $\{\varphi_n\}$ be an (α, β)-frame. We can expand Ψ as $\Psi = \Sigma_n c_n \varphi_n$. Writing $\Psi(t)$ as $\Psi(t) = e^{iHt}(\Sigma_n c_n \varphi_n) = \Sigma_n c_n(e^{iHt}\varphi_n)$, this equation says that also the set $\{e^{iHt}\varphi_n\}$ of t-depending vectors $e^{iHt}\varphi_n$ can be used to expand vectors of $\mathcal{H}$. This is not a surprise since $\{e^{iHt}\varphi_n\}$ is again an (α, β)-frame, as it is easily seen both with a direct computation and using Proposition 2. This result can be seen again as an evidence of time stability of frames of the same kind discussed in Section 3. In the particular case in which H satisfies (3.1) another frame can be generated starting from $\{\varphi_n\}$, which is essentially (but for the imaginary unit i) the time derivative of $e^{iHt}\varphi_n$ (computed for $t = 0$). It is not surprising that, as already seen in Subsection 3.2, there is in general no relation between the frame bounds of the two different frames.

In this example the generator operator is a unitary operator arising from the quantum evolution of a closed system. However, condition (4.1) is also satisfied trivially if T is simply an isometry, that is if $T^\dagger T = \mathbb{I}$ but $TT^\dagger \neq \mathbb{I}$. Of course isometries which are not unitary operators only exist in infinite-dimensional Hilbert spaces. We refer to Halmos (1967) for some examples.

5. OUTCOME AND FUTURE PLANS

We have seen how a class of bounded operators, self-adjoint or not, can be used to construct frames. This construction does not produce an unique result, since this depends on the o.n. basis or the frame which, in a certain sense, we are *perturbing*.

We have given several examples, some of them arising from quantum mechanical problem which show that the time evolution of both closed and open systems maps frames into frames.

What is still to be understood, in our opinion, is whether there exists a deeper relation between a frame generator and a frame, of the same kind of that which associates to any self-adjoint operator an unique (up to degenerations) o.n. basis made up with its eigenvectors.

Also, it would be interesting to understand if and how an o.n. basis can be extracted from a given (A, B)-frame, completing in this way the necessary condition given in the Lemma above.

Finally, we believe that the time stability of frames deserve a deeper investigation and that it would be rather interesting finding concrete physical applications.

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